Memory-flow structures in Lorenz chaos

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Memory-flow structures of a coupled three-element model, here a Lorenz chaos, have been investigated. Although in the models of coupled two-element systems, a bidirectional connectivity of information transference is held and the memory flows of one channel exhibit a nearly out-of-phase relation, it is found that in this Lorenz chaos one additional unidirectional connectivity can also occur and the nearly out-of-phase relation in the coupled two-element models will be replaced by an antiphaselike relation in Lorenz chaos.

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I. INTRODUCTION

It has been well known that the existence of chaos implies that the loss of memory of initial conditions is inherent in dynamic systems [1]. Usually the concept of loss of information can be quantitatively described by the Lyapunov exponent. However, the use of the Lyapunov exponent is limited by its characteristic nature of exponential divergence of close-by trajectories and it may not be useful for a long time scale [2]. To characterize such a memory loss of initial conditions, Gade and Amritkar [2] have proposed a generalized time-dependent exponent approach which is the same as the information theoretic approach of Fraser and Swinney [3], in terms of probability. In the approach of Fraser and Swinney, the information is calculated for the whole dynamic system because a time-delay coordinate attractor-reconstruction method has been adopted [3]. However, it is sometimes more interesting to investigate the spatial interplay of coupled-element systems (or spatially extended systems). In such cases, it is necessary to find the connection between two channels (or two spatial points). To calculate the mutual information flow between two channels, one possible solution is to follow the approach of Matsumoto and Tsuda [4] or Vastano and Swinney [5].

In terms of information theory, the mutual information with a time lag for a channel (i.e., self-information) characterizes the memory process embedded in the channel and the mutual information with a time lag between different channels characterizes the information transport structure between the channels. Two interesting questions are as follows: (i) How does a coupled-element system lose the memory of initial conditions due to the coupling between elements in chaos? (ii) What are the roles played by individual elements in information transport? The main concern is the interplay of the memory flows in chaos for the coupled-element systems. To inspect this topic, we have investigated different models of coupled two-element systems with time delays [6]. The basic features of memory flows have been explored. Based on these features, we are therefore able to construct a groundwork for a better understanding of the coupled three-element model, here a Lorenz chaos. We note these basic features for comparison: (i) the mutual information flows possess an almost in-phase relation between $M_{j,i}$ and $M_{i,j}$ $(i\neq j)$, i.e., a bidirectional connectivity of information transference always holds; (ii) the self-information and mutual information flows are almost out of phase, suggesting a process of information transport. This also suggests that there is a fundamental period inherent in the system such that the information among all elements has to be circulated in a period of time.

II. INFORMATION THEORETIC CHARACTERIZATION OF MEMORY FLOWS

For completeness, a brief survey of the information theoretic calculation used in this paper will be given. According to the calculation method of Ikeda and Matsumoto [7], we first calculate the self-information flow for the ith channel, i.e.,

$$S_i(\tau)=H_i(0)+H_i(\tau)-H_{i;\tau}(\tau)=2H_i(0)-H_{i;\tau}(\tau)\ , \qquad (1)$$

where $H_i=-\sum_{l=0}^{N-1}P_l\ln P_l$ is the Shannon entropy for the ith channel, within which P_l is the probability for the channel being in an l state among all possible N states generated by the ith channel, and the summation $\sum$ sums over all the states from $l=0$ to $N-1$. The joint Shannon entropy $H_{i;\tau}(\tau)=-\sum_{l,m}P_{l,m}(\tau)\ln P_{l,m}(\tau)$, where $P_{l,m}(\tau)$ means the probability in which an l state appears at time 0 (*initial conditions*) and an m state appears at time $0+\tau$ for the ith channel. We call τ the information flow time. In the simulation, we store long time series of the ith channel after a transient and determine the minimum, maximum, and output ranges. The output range is divided into N intervals by a unit ϵ, the limitation of resolution ability. The recorded data, after subtracting the minimum and dividing by ϵ, can be assigned to a state (from $l=0$ to $N-1$) according to the integer value. Because of finite ϵ, the information is coarse grained. In the simulation, we have adopted at least 40 000 events with adjusted resolution to ensure realizable P_l and $P_{l,m}$. $S_i(\tau)$ is the information generated by the ith channel at time $0+\tau$ in common with that at time 0. If at time τ the system completely loses this common information, i.e., the memory of the initial conditions, then $S_i(\tau)=0$. To investigate the information transport between two channels, we calculate the mutual information $M_{i,j}(\tau)$,

$$M_{i,j}(\tau)=H_i(0)+H_j(\tau)-H_{i,j}(\tau) , \quad (2)$$

which is the information generated by the jth channel at time $0+\tau$ in common with that of the ith channel at time 0. Thus it can be used to characterize information transport between the ith and jth channels with a time difference τ. In terms of memory, $M_{i,j}(\tau)$ is the amount of memory for the jth channel to remember the initial conditions of ith channel.

It is known that the mathematical knowledge of correlation functions is fairly well advanced. As emphasized by Gade and Amritkar [2], it can be shown that for a sine function, the autocorrelation function oscillates with time, giving no indication of this perfectly predictable system. On the other hand, the time-dependent generalized exponents (and so the self-information) remains constant with time, providing a perfectly predictable indication. Thus to investigate the memory-flow structures in dynamic systems, a correlation function is not a good tool (see Fig. 1 of Fraser and Swinney in Ref. [3]). Indeed, it has been pointed out by Li that there is an exact relation between the mutual information and the correlation function if the data are of binary sequences [8]. However, for sequences with more than two symbols, which often occurs to dynamic systems, no general relation has been derived.

III. MEMORY-FLOW STRUCTURES OF A LORENZ CHAOS

Now we consider a coupled three-elemental model, a Lorenz chaos. It is actually not difficult to imagine that there is a simple path of information transference in the coupled two-element systems: element 1(2)$\rightarrow$element 2(1)$\rightarrow$element 1(2). However, it is not simple at all to figure out the path formation even just for coupled *three* element system. More specific questions are as follows: (i) Should an "out-of-phase" of relation still exist? If not, what kind of constitution will take place? (ii) Should an "in-phase" relation still hold for $M_{i,j}$ and $M_{j,i}(i\neq j)$ flows? If not, what will occur? To answer these questions, we consider the well-known Lorenz equations as a prototypical model, i.e.,

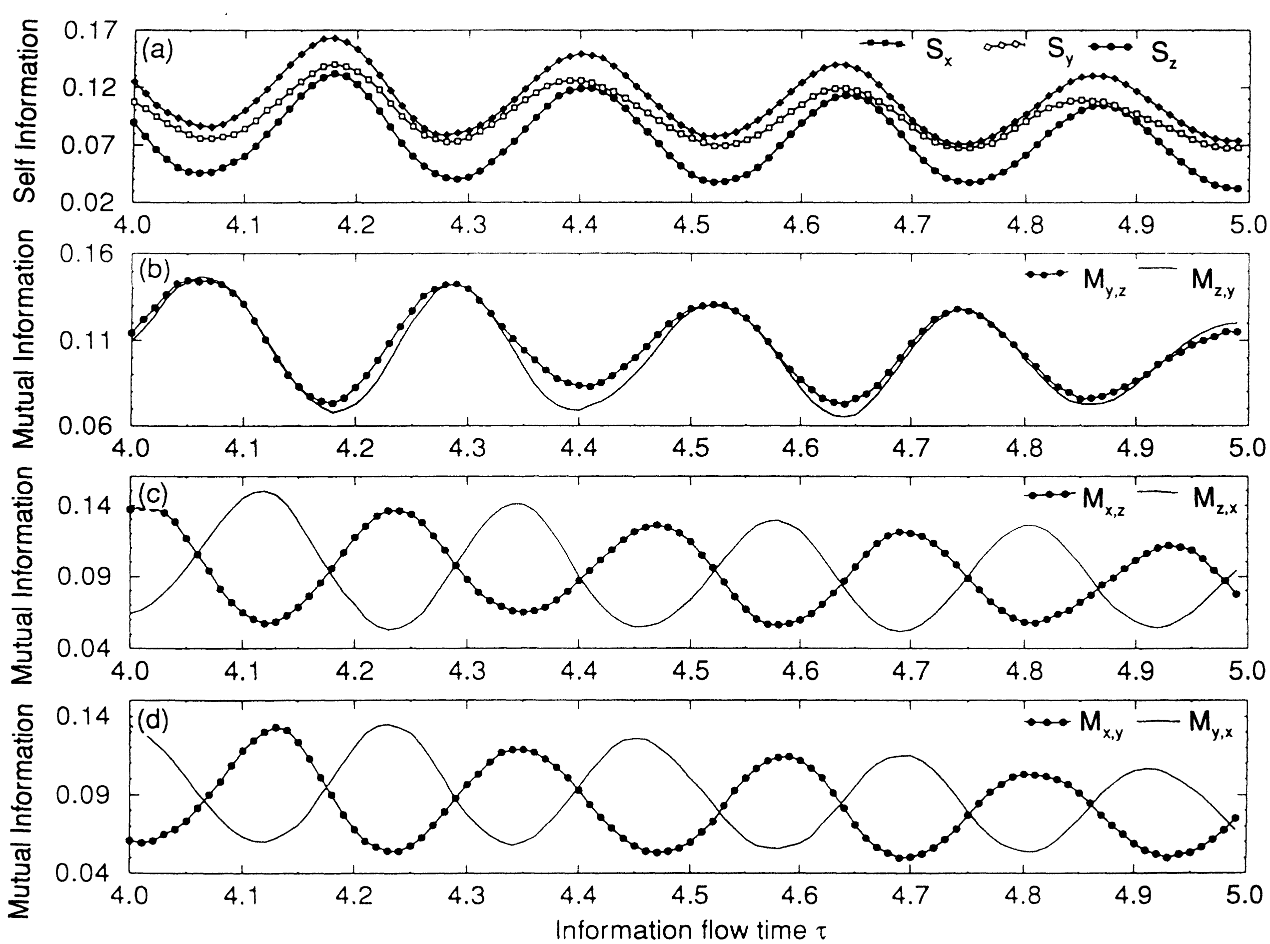

FIG. 1. Pair relation in the information flows of Lorenz chaos: (a) a nearly in-phase feature occurs in the self-information flows; (b) a nearly in-phase relation appears in the mutual information flows; (c) and (d) the out-of-phase features in the mutual information flows.

$$\frac{d}{dt}x(t)=-\sigma x(t)+\sigma y(t)\ , \tag{3}$$

$$\frac{d}{dt}y(t)=-x(t)z(t)+rx(t)-y(t)\ , \tag{4}$$

$$\frac{d}{dt}z(t)=x(t)y(t)-bz(t)\ , \tag{5}$$

where x is proportional to the intensity of the convective motion, y is proportional to the temperature difference between the ascending and descending currents, and z is proportional to the distortion of the vertical temperature profile from linearity. Furthermore t is a normalized time, σ is the Prandtl number, b is a constant related to the aspect ratio, and r is related to the external control parameter. The details can be found in Ref. [9]. In this paper, we consider the case in which $\sigma=15$ and $b=4$. Under these conditions, the critical value of r is $r_c=33$. We set $r=45.92$ to drive the system into a chaotic state. It is noted that the time series $y(t)$ is similar to $x(t)$. We will show later that the value of the simultaneous mutual information between x and y is indeed considerably large. However, the dynamic evolution of the mutual information cannot be seen just by a direct comparison between two time series. Since we can take approximately a difference scheme in place of the differential term, e.g., $dx(t)/dt\approx[x(t)-x(t-\Delta t)]/\Delta t\to x(n)-x(n-1)$, a simple coupling configuration for Lorenz equations is available. It is straightforward to derive the coupling configuration for coupled two-element models. Bidirectional connectivity of information transference in coupled two-element models thus seems expectable, based on the appearance of self-coupling and bidirectional coupling shown in configuration. This is not true for the Lorenz chaos, however, as we will show later.

Now we examine the memory flows for such a Lorenz chaos. It is shown in Fig. 1(a) that all three self-information flows present similar evolutions, even though the time series are different. We can classify the mutual information in pairs according to the phase relation. As shown in Fig. 1(b), $M_{y,z}$ and $M_{z,y}$ evolve with almost the same phase, which is similar to what occurred in coupled two-element systems [6]. Recall that the in-phase relation implies the occurrence of a bidirectional connectivity. Nevertheless, this almost in-phase relation is broken

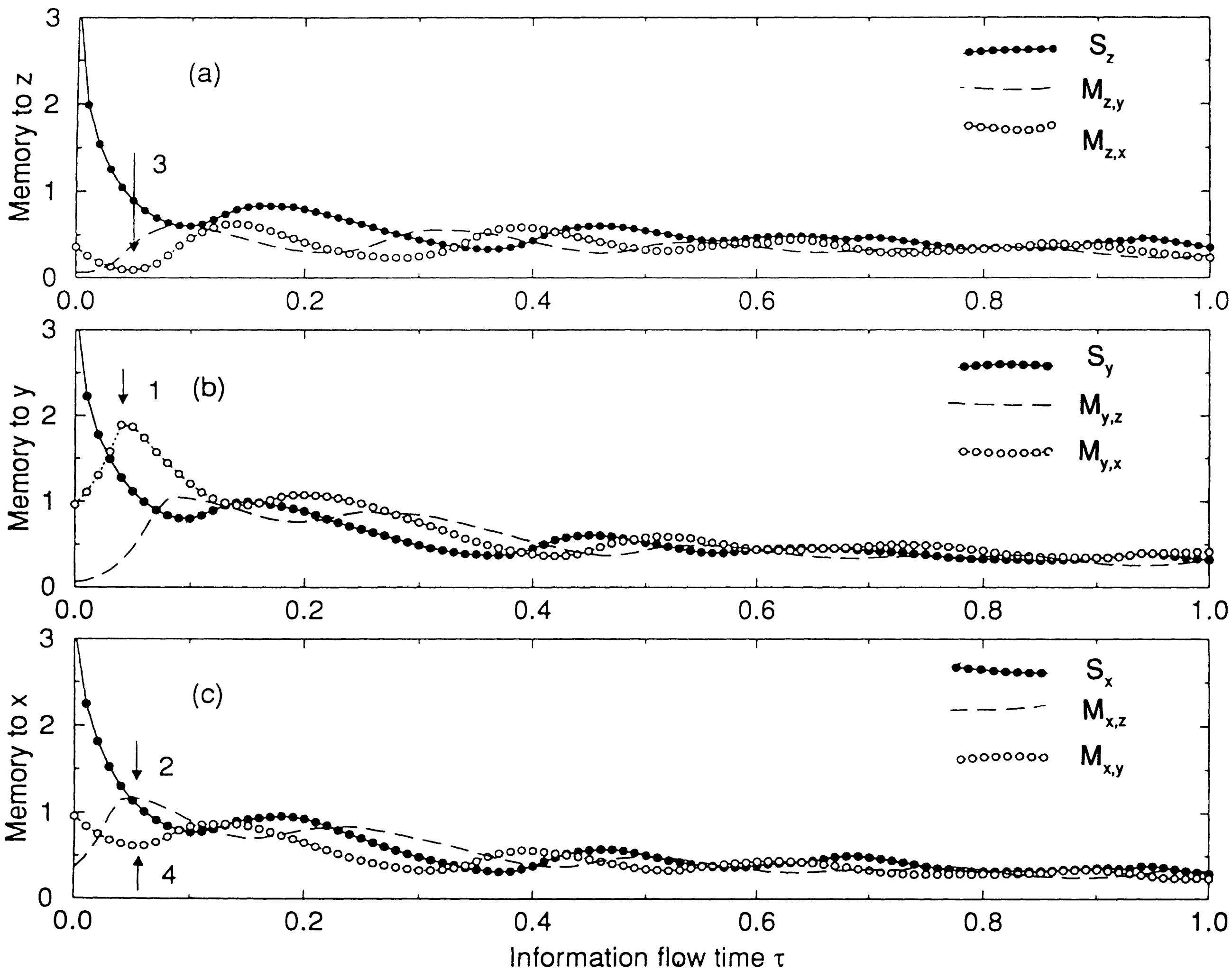

FIG. 2. Memory flows for the channels x, y, and z in a Lorenz chaos.

in cases of $M_{z,x}$ and $M_{x,z}$ (also $M_{x,y}$ and $M_{y,x}$) since they are actually almost out of phase, as shown in Figs. 1(c) and 1(d). (Note that $x(t)$ looks like $y(t)$, resulting in a high $M_{x,y}(0)$ [$=MI_{y,x}(0)$], as shown in Figs. 2(b) and 2(c).) This means that in Lorenz chaos, the connectivity of information transference does not have to be bidirectional (even though the variables are mutually coupled). That is why, for example, x can "remember" more about the initial conditions of y, but at the same time y loses more memory about the initial conditions of x. This feature is amazing because it implies that terms such as "selfish" and "nonmutually beneficial" still may be applicable to nonbiological systems. Here "selfish" can have a solid definition based on the transport character of information.

To characterize the evolution process of information transference in detail, we can label the top and bottom points of memory flows as the "turning" points. As an example, we look at the evolution from $\tau=0$ to τ_1, the first turning point (indicated by $\downarrow 1$). Referring to Fig. 2(b), the memory of every initial condition of any specific channel can be interpreted as follows.

(i) Since S_z decreases, the z channel loses the memory of its own initial condition. Meanwhile, because $M_{z,x}$ also decreases, x also loses the memory of the initial condition of z. Nevertheless, $M_{z,y}$ increases at the same time. This suggests that information is transferred from z to y.

(ii) Because S_y decreases, channel y loses its own memory. But M_{yx} increases coincidently (as does M_{yz}). This means that information is transferred from channel y to channels x and z.

(iii) Again, the x channel loses its own memory due to the decrease of S_x. In addition, y also loses its memory of the initial conditions of x. However, M_{xz} increases. This indicates that information has been transferred from x to z.

By this method, one can numerically determine the whole evolution process of information transference. To study the transition in memory flows, a fast Fourier transform spectrum analysis has been taken. We have found that there are obvious differences in different evolution stages. For comparison, we also have calculated the total information of the whole system and some different characteristics are found in the memory flows of the whole system.

IV. CONCLUSIONS

In summary, a Lorenz chaos has been considered as a prototypical model to investigate the information transference between different channels of a coupled three-element model. The above states that there is a unidirectional information transference which cannot be read from the coupling configuration and time series. It is worthwhile to emphasize that the unidirectional connectivity is a nonmutually beneficial relation. It has the sense of "*selfishness.*"

Let us summarize the general feature of this coupled three-element model in comparison with those of a coupled two-element model. It is true that information transference and memory flow are information dissipative in nature due to chaos. The direction of information transference does not have to be bidirectional. Unidirectional transference actually occurs more often, as observed in the Lorenz model. In regard to the out-of-phase feature presented in the memory flows of one channel in the coupled two-element models, we should notice that the out-of-phase feature has been replaced by an almost antiphase relationship characteristic of Lorenz chaos. This suggests that information must be circulated among all coupled elements in a fundamental period of time.

Our results on the direction of information transference may provide an interesting issue which is worthy of further study, particularly for the biological systems. It is certainly true that our present study is still strongly model dependent. A question arises as to what is the interplay between the memory flows for other models. Particularly, is it possible to have some universal and quantitative features for the memory flow structure in coupled element models, such as neural networks or brain models? We leave these interesting issues to future works.

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